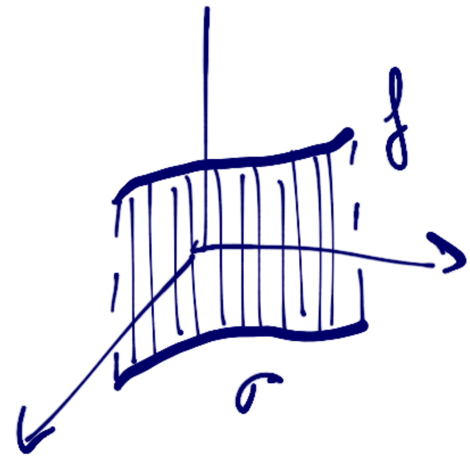
Line integrals

$\int_\sigma f \equiv$ average of density f along the trajectory
line
path

$f: \mathbb{R}^n \longrightarrow \mathbb{R}$

$\sigma \equiv$ trajectory.



$\int_\sigma F \equiv$ represents how a particle moves along the trajectory under the action of F

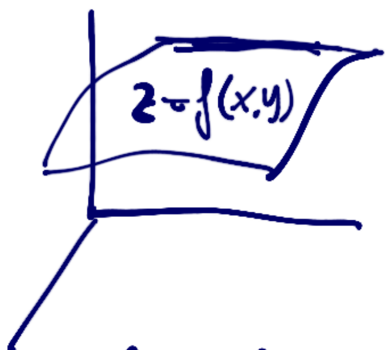
$F: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ vector field.

Surface integrals

Integrals of scalar and vector fields on surfaces.

Surface could be the graph of a scalar function.

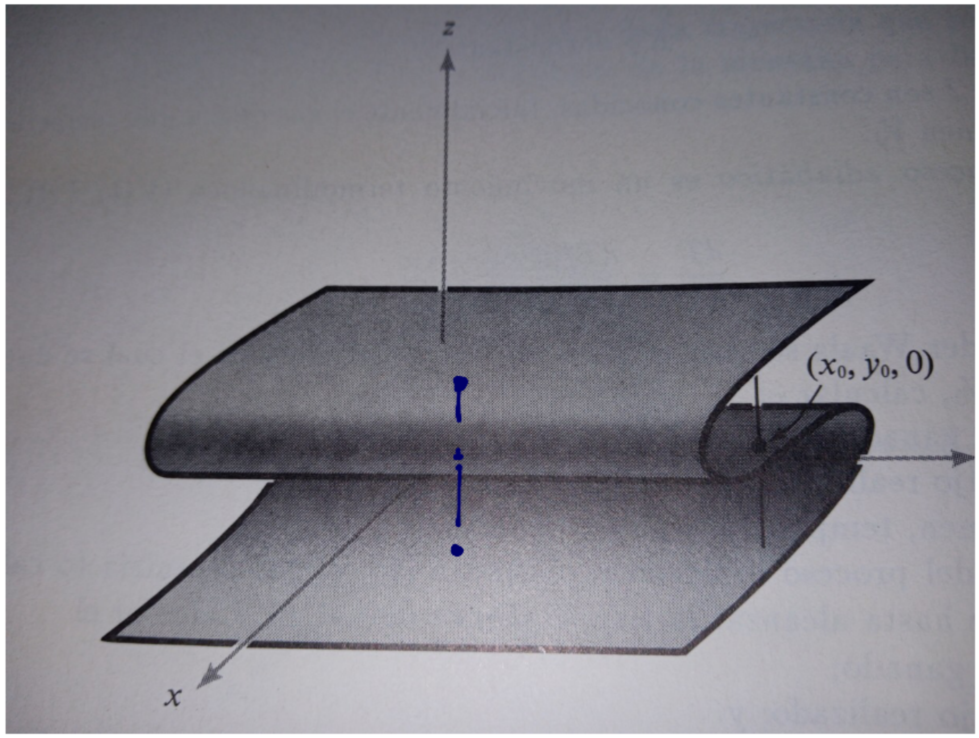
$$z = f(x,y)$$



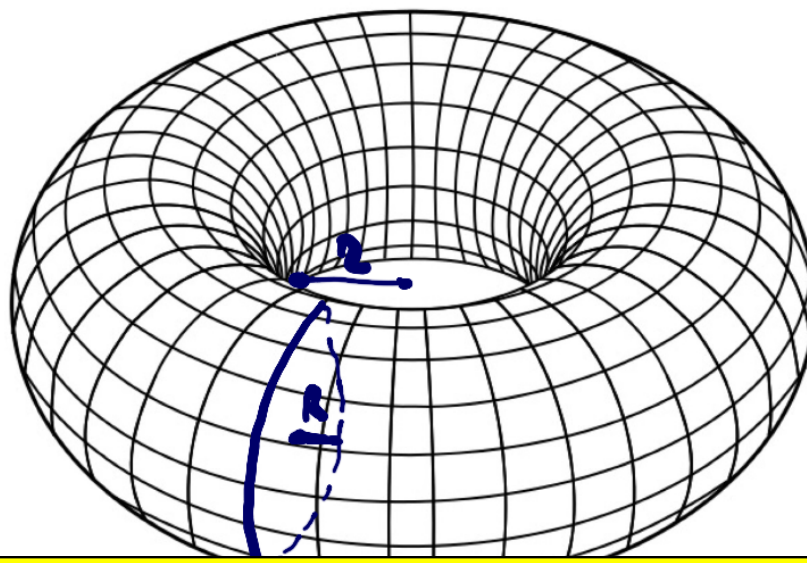
we will consider more general surfaces.
not only the graph of function.

$x - z + z^3 = 0$

(Bended paper)



Famous Torus



a function / as a parametrization

$\neq$ its image (geometrical object)

<u>Definition</u>

A parametrised surface is a continuous function

$$\Phi : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

with $D \equiv$ parametrical domain in $\mathbb{R}^2$

Then, the surface S is the image of the function ϕ, $S = \Phi(D)$ and we can write

$$\Phi(u,v) = \big(x(u,v),\ y(u,v),\ z(u,v)\big)$$

Remark

- $S = \Phi(D)$ is simple if the function Φ is one-to-one.

- Φ = parametrization. is differentiable or of class C^1, then the surface is C^1

(smooth surface)

Example: Plane $x+y+z=1$: Implicit form/surface

$$\left.\begin{aligned} x(u,v) &= x = u \\ y(u,v) &= y = v \\ z(u,v) &= z = 1-u-v \end{aligned}\right\}$$ Parametrisation of the surface.

Definition

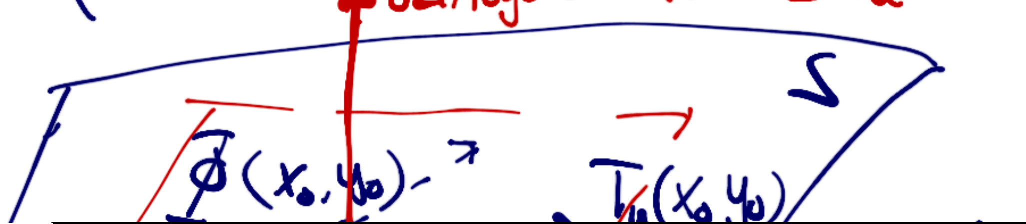
For a parametrised surface $S = \Phi(D)$, smooth so Φ diff. at any point in D we define the tangent vectors or tensors.

$$T_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) = \frac{\partial \Phi}{\partial u}$$

and

$$T_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right) = \frac{\partial \Phi}{\partial v}$$

orthogonal vector $= T_u \times T_v$

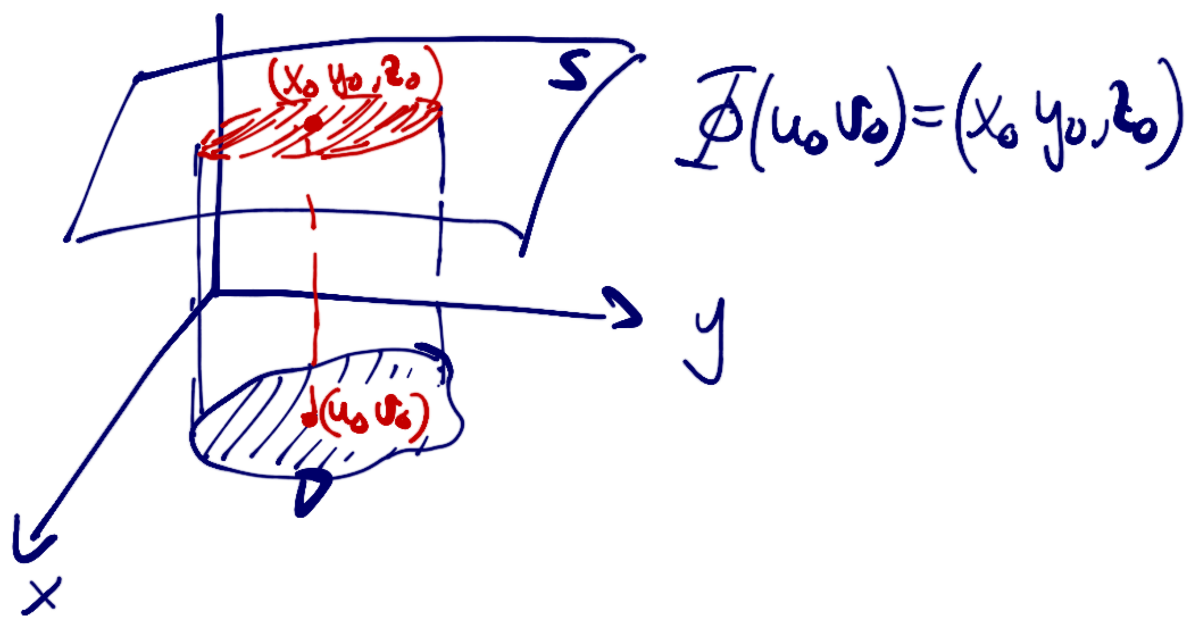


Definition

Let $\Phi : D \to \mathbb{R}^3$ be a regular/smooth surface

$$S = \Phi(D) \qquad \Downarrow \quad \phi \in C^1$$

and $T_u \times T_v \neq 0$ at any point in D



$$(T_u \times T_v)(u_0, v_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

$\| T_u \times T_v \|$

Example: Parametrise the sphere (Problem 1 i) in 4.3)

$$S = \{(x,y,z) \in \mathbb{R}^3, \ x^2+y^2+z^2 = R^2\}$$

Use the function $\Phi : D \longrightarrow \mathbb{R}^3$

$$\Phi(u,v) = \big(\Phi_1(u,v), \Phi_2(u,v), \Phi_3(u,v)\big)$$

applying spherical coordinates. (with fixed radius R)

$$\left.\begin{aligned} x &= R\cos u \sin v \\ y &= R \sin u \sin v \\ z &= R \cos v \end{aligned}\right\} \qquad \begin{aligned} &D \subset \mathbb{R}^2 \\ &(u,v) \in D \end{aligned}$$

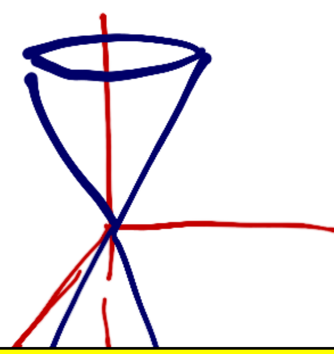
Different parametrization.

$$\Phi_{\pm}(u,v) = \left(u, v, \pm\sqrt{R^2-u^2-v^2}\right)$$ not one-to-one

Example: Problem 1-ii) in 4.3

Parametrise a cone

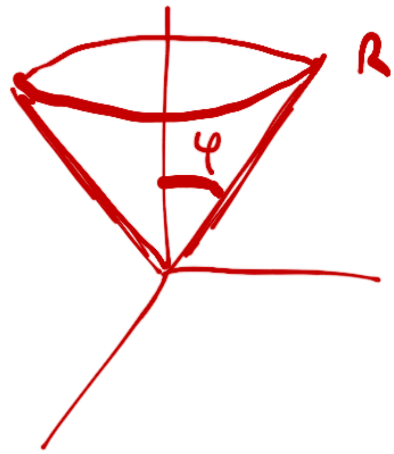
$$S = \{(x,y,z) \in \mathbb{R}^3 , \; x^2+y^2 = a z^2\}$$



$$\Phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$$

$$(u,v) \to \Phi(u,v)$$

Use again spherical coordinates.



$$\left.\begin{aligned} x &= \rho \cos u \sin v \\ y &= \rho \sin u \sin v \\ z &= \rho \cos v \end{aligned}\right\}$$

Substituting into the expression of S.

$$\underbrace{x^2 + y^2}_{\parallel} = a z^2$$

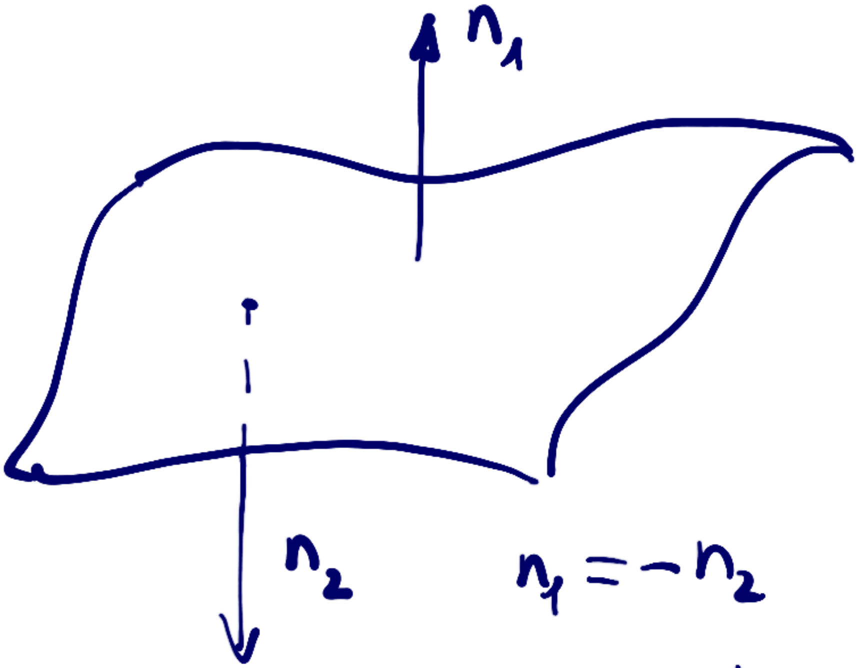
$$\sin^2 v = a \cos^2 v \Rightarrow \underline{\underline{\operatorname{tg} v = \pm\sqrt{a}}}$$

$$v = \pm \operatorname{arctg}\sqrt{a}$$

$$v = \operatorname{arctg}\sqrt{a}$$

Orientation

Exterior or interior orientation



$n_1 = -n_2$

Paradigmatic example: **Moebius strip.**
does not have orientation.

Example: $\underbrace{x^2+y^2+z^2=1}_{S \text{ in } \mathbb{R}^3}$: Sphere of radius 1.

Parametrization:

$$\left.\begin{aligned} x &= \cos\theta \sin\varphi \\ y &= \sin\theta \sin\varphi \\ z &= \cos\varphi \end{aligned}\right\} \qquad \begin{aligned} &\Phi : D \to \mathbb{R}^3 \\ &D = \left\{ (\theta,\varphi) \in \mathbb{R}^2,\ \begin{aligned} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \end{aligned} \right\} \end{aligned}$$

$$\Phi(\theta,\varphi) = (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi)$$

Tangent vectors.

$$T_\theta = \frac{\partial \Phi}{\partial \theta} = (-\sin\theta \sin\varphi, \cos\theta \sin\varphi, 0)$$

Orientation is given by the orthogonal vector.

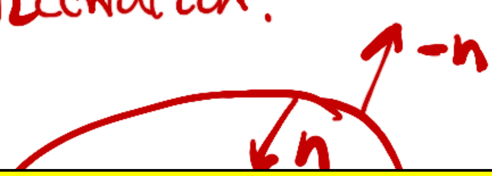
$$T_\theta \times T_\varphi = \begin{vmatrix} i & j & k \\ -\sin\theta\sin\varphi & \cos\theta\sin\varphi & 0 \\ \cos\theta\cos\varphi & \sin\theta\cos\varphi & -\sin\varphi \end{vmatrix}$$

$$= -\sin\varphi$$

Since $0 \leq \varphi \leq \pi$

$$-\sin\varphi \leq 0$$

then the normal vector points inward changing the orientation.



Surface Integrals.

Definition

$\underline{\Phi}: D \to \mathbb{R}^3$ parametrization of a differentiable surface $S = \underline{\Phi}(D)$ and

$f: \underline{\Phi}(D) \to \mathbb{R}$ cont. scalar function

Then

$$\boxed{\int_{\underline{\Phi}} f \cdot dS = \iint_D f(\underline{\Phi}(D)) \cdot \|T_u \times T_v\| \, du \, dv}$$

In particular, the area of a surface S

average value of f in $S = \Phi(D)$ as

$$\boxed{\frac{1}{A(S)} \int_{\Phi} f \, dS.}$$

Example: $S : x^2+y^2+z^2=1$ Problem 1 i) - 4.3

$$\Phi : \left.\begin{array}{l} x = \cos\theta \sin\varphi \\ y = \sin\theta \sin\varphi \\ z = \cos\varphi \end{array}\right\} \quad D = \{ 0 \le \theta \le 2\pi, \ 0 \le \varphi \le \pi \}$$

$\Phi : D \to \mathbb{R}^3$, $\| T_\theta \times T_\varphi \| = \sin\varphi$

Then

$\int_0^{2\pi} \int_0^{\pi}$

Remark

- $\int_{\Phi_1} f \cdot ds = \int_{\Phi_2} f \cdot ds$ two parametrizations Φ_1, Φ_2

- Area of a surface S which is the graph of a function of class C^1

$$z = g(x, y)$$

It might be parametrized by

$x = u$ (: Φ then

$y = v$

Definition

$\Phi : D \to \mathbb{R}^3$ diff (of class C^1)

$F : S = \Phi(D) \to \mathbb{R}^3$ vector field.

$$\int_{\Phi} F \cdot ds = \iint_D F(\Phi(D)) \cdot \underline{\underline{(T_u \times T_v)}}\, du\, dv$$

However, for two parametrizations. Φ_1 and Φ_2

a) $\int_{\Phi_1} F = \int_{\Phi_2} F$ with the same orientation

Theorems relating differential vectorial calculus and integral vectorial calculus.

$\Downarrow$

arise in physical problems.

Green's Theorem

It establishes a relation between a line integral along a closed curve and a double integral on the enclosed region.

Green's Theorem (in $\mathbb{R}^2$)

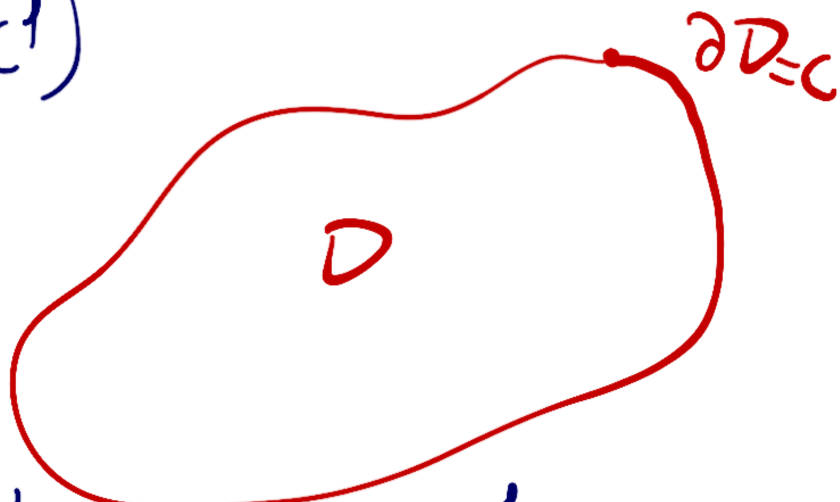
$D \subset \mathbb{R}^2$ open set in the plane

$\partial D \equiv$ boundary of D is closed, simple

$\sigma(a)=\sigma(b)$ no multiple points.

and smooth

(of class C^1)



$\sigma \equiv$ parametric form of

$\partial D = C$

P, Q two scalar functions of class C^1

we find that

$$\underbrace{\int_{\partial D} F\cdot dr}_{\text{line integral.}} \quad = \quad \underbrace{\iint_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy}_{\text{double integral.}}$$

Remark

a) Orientation

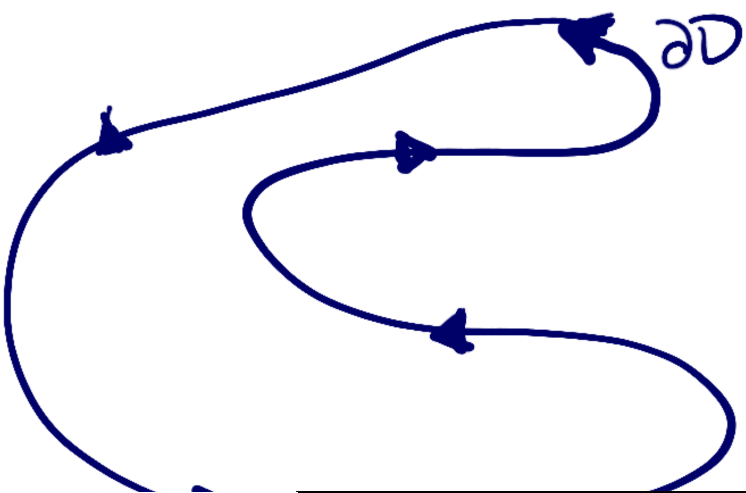
Positive orientation $\rightsquigarrow$ anticlockwise.

b) Curve must be simple.

σ must be one-to-one.

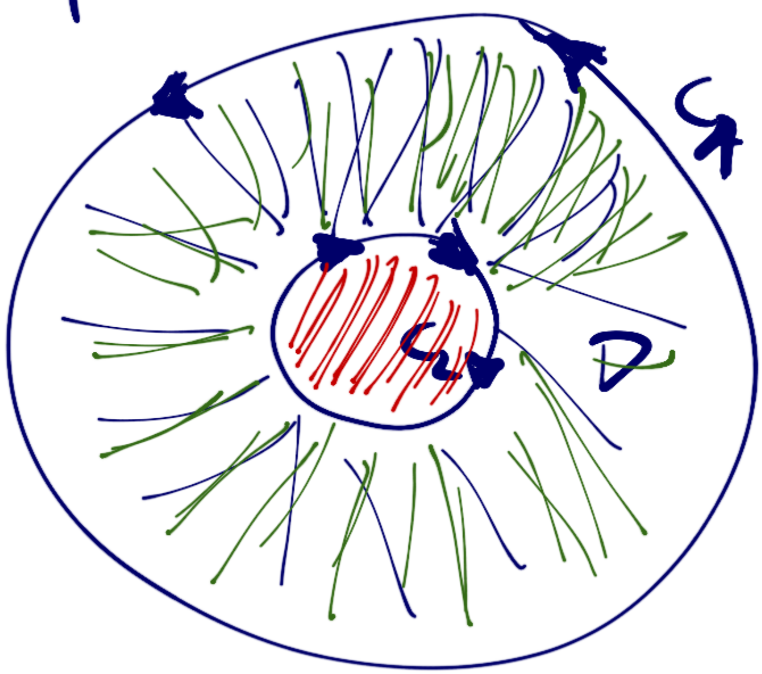
c) Generalisation of Green's theorem

For general regions.



For some regions we must perform some transformations to apply the theorem.

For example an annulus



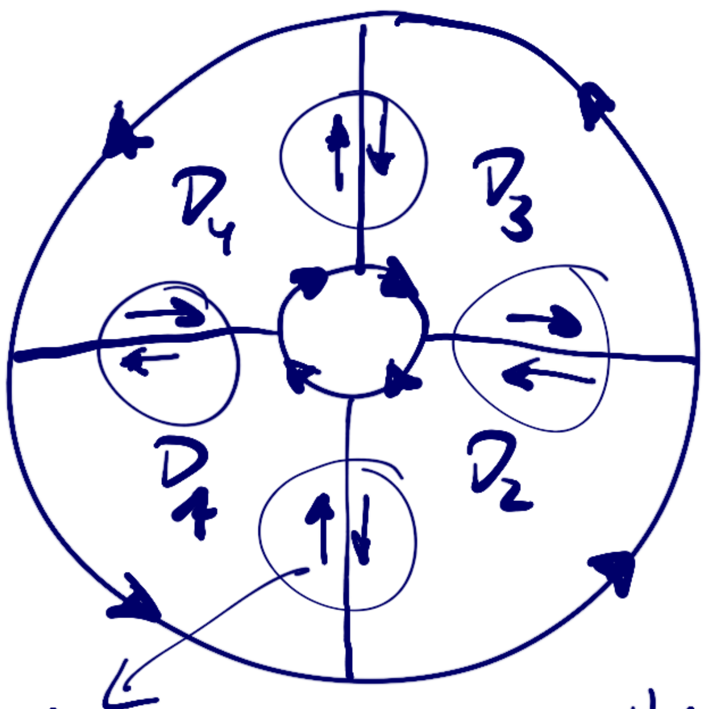
$C = C_1 \cup C_2$

$$\int_C F \cdot dr = \underbrace{\int_{C_1} F \cdot dr}_{} + \underbrace{\int_{C_2} F \cdot dr}_{}.$$

How is the boundary?

$\Downarrow$

To apply Green's Theorem we can divide the region.



$D = \underline{\underline{D_1 \cup D_2 \cup D_3 \cup D_4}}$

line integrals for those artificial lines cancel each other.

We can apply Green's Theorem for

Example:

$$I = \int_{C} (x+2y)\,dx + (3x-y)\,dy$$

$$F(x,y) = (\underbrace{x+2y}_{P(x,y)}, \underbrace{3x-y}_{Q(x,y)})$$

$C = \{ (x,y) \in \mathbb{R}^2 , \ x^2 + 4y^2 = 4 \}$ ellipse.

a) Parametrization

$$\left.\begin{array}{l} x = 2\cos t \\ y = \sin t \end{array}\right\} \quad t \in [0, 2\pi] , \quad \sigma(t) = (2\cos t, \sin t)$$

$$\int^{2\pi}$$

$$\sigma'(t) = (-2\sin t, \cos t)$$

$$I = \int_0^{2\pi} \left(-5\cos t \sin t - 4\underbrace{\sin^2 t}_{\frac{1-\cos 2t}{2}} - 6\underbrace{\cos^2 t}_{\frac{1+\cos 2t}{2}}\right) dt$$

$$= 2\pi$$

6) Green's Theorem (quicker)

$$\underline{\underline{I}} = \int_C F \cdot dr = \iint_E \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

$$= \iint_E (3-2)\, dx\, dy = \underline{\text{Area of Ellipse.}}$$